# Expansion of Gas Clouds and Hypersonic Jets Bounded by a Vacuum

H. Mirels* and J. F. Mullen†
*Aerospace Corporation, El Segundo, Calif.*

**The unsteady expansion of gas clouds, bounded by a vacuum, is considered for both continuum and free molecular flows. Known analytical solutions are reviewed with emphasis on the asymptotic nature of the flow after long times. An approximate analytical solution is presented for the continuum expansion, after long time, of an initially uniform gas cloud. For a plane expansion, the approximate solution is in exact agreement with the known exact solution. For cylindrical and spherical expansions, the approximate solution appears to give results for density (at the axis) which are higher than those obtained by a numerical integration of the equations of motion. Insufficient data are available for a precise evaluation of the approximate solution. The continuum and free molecular expansions of an initially uniform gas are found to be quite similar, after long time, except for the flow in the vicinity of the leading edge of the expansions. For a given gas, the density at the axis decays more rapidly, with time, for the continuum case. The extension of these results to hypersonic jets, bounded by a vacuum, also is noted.**

## Nomenclature

$a$ = sound speed
$F$ = hypergeometric function
$h$ = $\gamma/2a_0^2$
$M$ = Mach number
$m$ = elemental mass
$N$ = $(\gamma + 1)/2(\gamma - 1)$
$p$ = pressure
$R$ = ordinate of leading edge of expansion
$r$ = spatial coordinate, transverse for jet flows
$t$ = time
$u$ = axial velocity for jet flow
$v$ = velocity, transverse for jet flows
$x$ = axial coordinate for jet flows
$\alpha,\beta$ = characteristic Riemann invariants
$\Gamma$ = gamma function
$\gamma$ = specific heat ratio
$\delta$ = characteristic streamline slope
$\eta$ = similarity variable
$\rho$ = density
$\sigma$ = dimensional index (0, 1, 2)

## Introduction

SPACE flight has intensified interest in the unsteady expansion of gas clouds into a vacuum. A related problem, the lateral expansion of a hypersonic jet bounded by a vacuum, also is of current interest.

Several analytic solutions exist for unsteady expansions into a vacuum. References 1 and 2 have treated plane, cylindrical, and spherical self-similar flows. These are flows where time can be eliminated from the equations of motion, reducing them to ordinary differential equations that can be integrated in closed form. Only special initial conditions for a gas cloud can lead to self-similar solutions. In particular, the initial density distribution cannot be uniform. The plane problem of an initially uniform gas cloud, expanding into a vacuum, has been solved analytically in Ref. 3. The solution for an initially uniform cylindrical or spherical gas cloud requires numerical integration of the equations of motion. The foregoing studies use continuum equations of motion. If the mean free path in a gas cloud is considerably larger than the cloud diameter, the resulting expansion can be treated as a free molecule flow. Reference 4 has obtained solutions, in quadrature form, for the free molecular expansion of arbitrary gas clouds.

In the present study, the asymptotic nature of the flow field, after long times, is examined for each of the analytical solutions mentioned in the foregoing. These are used as a guide to develop an approximate asymptotic solution for the expansion of an initially uniform cylindrical or spherical gas cloud. Reference 5 has discussed the asymptotic nature of the flow, near the origin, for the plane problem of an initially uniform gas cloud expanding into a vacuum. The present study is a generalization of Ref. 5 in that the entire flow field between the origin and the leading edge of the expansion is considered. In addition, cylindrical and spherical expansions are considered. The application of the unsteady plane and cylindrical flow results for finding the lateral spreading of a steady hypersonic jet, bounded by a vacuum, also is discussed.

Received by ARS October 11, 1962; revision received January 28, 1963. This paper was prepared under Contract No. AF 04(695)-169.

* Head, Advanced Propulsion and Fluid Mechanics Department, Laboratories Division. Member AIAA.
† Member of the Technical Staff, Laboratories Division.

## Unsteady Expansion of Gas into Vacuum

### General Considerations

The equations of motion for unsteady flows in one spatial variable $\bar{r}$ may be written as follows:

Continuity

$$\frac{\partial\bar{\rho}}{\partial\bar{t}} + \frac{\partial\bar{\rho}\bar{v}}{\partial\bar{r}} + \sigma\,\frac{\bar{\rho}\bar{v}}{\bar{r}} = 0 \tag{1a}$$

Momentum

$$\frac{\partial\bar{v}}{\partial\bar{t}} + \bar{v}\,\frac{\partial\bar{v}}{\partial\bar{r}} = -\,\frac{1}{\bar{\rho}}\,\frac{\partial\bar{p}}{\partial\bar{r}} = -\gamma c\bar{\rho}^{(\gamma-2)}\,\frac{\partial\bar{\rho}}{\partial\bar{r}} \tag{1b}$$

where the superscript bar indicates a dimensional quantity and $\sigma = 0, 1, 2$ for plane, cylindrical, and spherical flows, respectively. See the Nomenclature for definition of the other variables. *Equation (1b) and all subsequent developments assume that the entropy of the gas is uniform* [i.e., the relation $\bar{p} = c\bar{\rho}^{\gamma}$, where $c$ is a constant, has been used to eliminate the variable $p$ in Eq. (1b) so that there are now two dependent variables, $\bar{\rho}$ and $\bar{v}$].

Equations (1) can be nondimensionalized by introducing the following variables:

$$\rho = \frac{\bar{\rho}}{\bar{\rho}_0} \qquad v = \frac{\bar{v}}{\bar{a}_0} \qquad r = \frac{\bar{r}}{\bar{R}_0} \qquad t = \frac{\bar{t}\bar{a}_0}{\bar{R}_0} \tag{2}$$

In all subsequent developments, $\bar{R}_0$ will be taken to be the initial ($\bar{t} = 0$) gas cloud radius. For initially uniform gases, $\bar{\rho}_0$ and $\bar{a}_0$ are the initial density and sound speed, respectively. For initially nonuniform gases, $\bar{\rho}_0$ and $\bar{a}_0$ are evaluated at $\bar{r} = 0$, $\bar{t} = 0$. Substitution of Eqs. (2) into Eqs. (1) yields

$$(\partial\rho/\partial t) + (\partial\rho v/\partial r) + \sigma(\rho v/r) = 0 \qquad (3a)$$

$$(\partial v/\partial t) + v(\partial v/\partial r) = -\rho^{\gamma-2}(\partial\rho/\partial r) \qquad (3b)$$

which are the nondimensional equations considered herein.

As pointed out in Ref. 6, the pressure gradient term in the momentum equation [right side of Eq. (3b)] becomes negligible, after long times, when studying the expansion of gases into a vacuum. Equation (3b) then can be integrated to yield

$$v = r/t \qquad (4)$$

Reference 6 refers to such flows as "inertia dominated." That is, each gas particle moves with a constant velocity and is unaffected by the other particles. Substitution of Eq. (4) into Eq. (3a) shows that the density distribution must have the form

$$\rho t^{\sigma+1} = f(r/t) \qquad (5)$$

where $f(r/t)$ is a function of $r/t$. The precise form for this function depends on the earlier motion of the gas cloud. The motion is self-similar, after long times, since $v$ and $\rho t^{\sigma+1}$ are functions only of $r/t$. All solutions discussed herein will have this asymptotic behavior.

**Self-Similar Flows**

References 1 and 2 have investigated expansions into a vacuum which are self-similar at all times. These flows now will be discussed.

Let $R(t)$ be the leading edge of a gas cloud that is expanding into a vacuum. The gas particles then are contained in the region $0 \leq r \leq R(t)$. A vacuum exists for $r > R(t)$. Define a similarity variable

$$\eta = r/R(t) \qquad (6)$$

so that $\eta = 1$ corresponds to the leading edge of the gas cloud and the gas is confined to the region $0 \leq \eta \leq 1$. Self-similar solutions are found by assuming that the dependent variables have the form $v = g(t)\phi(\eta)$ and $\rho = h(t)f(\eta)$, substituting into Eqs. (3), and finding the form of $g(t)$ and $h(t)$ such that the resulting equations are independent of $t$. The resulting solution, for a gas expanding into a vacuum,[1, 2]

$$\rho = R^{-(\sigma+1)}[1 - \eta^2]^{1/(\gamma-1)} \qquad (7a)$$

$$v = (dR/dt)\eta \qquad (7b)$$

where

$$\frac{dR}{dt} = \frac{1}{(\sigma+1)^{1/2}}\,\frac{2}{\gamma-1}\,[1 - R^{-(\sigma+1)(\gamma-1)}]^{1/2} \qquad (8)$$

This solution satisfies the boundary conditions:

| | | | | |
|---|---|---|---|---|
| at | $t = 0$; | $R = 1$ | $dR/dt = 0$ | (9a) |
| | $t = 0$ | $r = 0$; | $\rho = 1$ | (9b) |
| | $t \geq 0$ | $\eta = 1$; | $\rho = 0$ | (9c) |
| | $t \geq 0$ | $\eta = 0$; | $v = 0$ | (9d) |

Thus, initially ($t = 0$), $v = 0$, and $\rho$ varies from 1 at $r = 0$ to 0 at $r = R$. At later times, $v$ is not zero and varies linearly with $r$. The variation of $v/(dR/dt)$ and $\rho R^{\sigma+1}$ with $\eta$ is illustrated in Fig. 1.

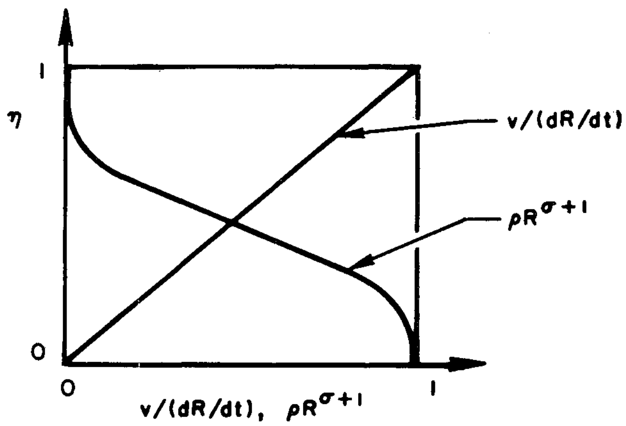


**Fig. 1 Variation of $v/(dR/dt)$ and $\rho R^{\sigma+1}$ with $\eta$**

The location of the leading edge of the expansion, at any time, can be found by integrating Eq. (8) which can be put in the form

$$\frac{1}{(\sigma+1)^{1/2}}\,\frac{2}{\gamma-1}\,t = \int_1^R \frac{dR}{[1 - R^{-(\sigma+1)(\gamma-1)}]^{1/2}} \qquad (10a)$$

$$= N_0 \int_0^{\sec^{-1}R^{1/N_0}} (\sec\theta)^{1+N_0}\,d\theta \qquad (10b)$$

where $N_0 \equiv 2/(\sigma+1)(\gamma-1)$. Equation (10b) is obtained from Eq. (10a) by letting $R = \sec^{N_0}\theta$. Equation (10b) can be integrated in closed form when $N_0$ is an integer. For example,

$$\frac{1}{(\sigma+1)^{1/2}}\,\frac{2}{\gamma-1}\,t = (R^2 - 1)^{1/2} \qquad N_0 = 1 \quad (11a)$$

$$= [R(R-1)]^{1/2} + \ln[(R-1)^{1/2} + (R)^{1/2}] \qquad N_0 = 2 \quad (11b)$$

$$= (R^{2/3} + 2)(R^{2/3} - 1)^{1/2} \qquad N_0 = 3 \quad (11c)$$

Equation (11a) includes the case $\sigma = 2$, $\gamma = \frac{5}{3}$. Equation (11b) includes the cases $\sigma = 1$, $\gamma = \frac{3}{2}$ and $\sigma = 2$, $\gamma = \frac{4}{3}$. Equation (11c) includes the cases $\sigma = 0$, $\gamma = \frac{5}{3}$; $\sigma = 1$, $\gamma = \frac{4}{3}$; and $\sigma = 2$, $\gamma = \frac{11}{9}$. Thus Eq. (10b) permits the complete closed form solution for a large number of cases of interest. The reduction of Eq. (10a) to Eq. (10b) does not appear to have been presented previously. The variation of $R$ with $t$ is illustrated in Fig. 2, which presents results for $\gamma = \frac{4}{3}, \frac{5}{3}$ and $\sigma = 0$, 1, and 2.

The asymptotic nature of the flow, after long times, now will be noted. Equation (10a) shows that for $R^{-(\sigma+1)(\gamma-1)} \ll 1$ (i.e., after long times),

$$R = \frac{1}{(\sigma+1)^{1/2}}\,\frac{2}{\gamma-1}\,t \qquad (12)$$

Thus, after long times, the leading edge moves with a uniform velocity $dR/dt = 2/[(\sigma+1)^{1/2}(\gamma-1)]$. For $\sigma = 0$, this velocity is the same as that associated with the expansion of an *initially uniform* gas cloud into a vacuum.‡ However, for $\sigma = 1$, 2, the final velocity is less, by the factor $1/(\sigma+1)^{1/2}$, than the corresponding velocity for an initially uniform

‡ For an initially uniform gas cloud, $dR/dt = 2/(\gamma-1)$ for all $t > 0$, regardless of the value of $\sigma$. (The case of an initially uniform gas and $\sigma = 0$ is treated in the next section.) This is true because at $t = 0$ the flow in the vicinity of $r = R$ behaves as though $\sigma = 0$. The leading edge achieves the limiting velocity $dR/dt = 2/(\gamma-1)$ and is unaffected by the subsequent expansion of the remainder of the gas. See Ref. 7 for further discussion.

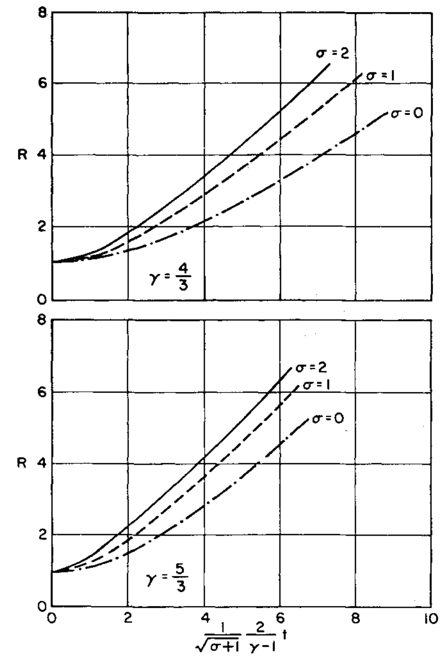


**Fig. 2 Leading edge of expansion for self-similar flows (from Ref. 1, p. 276)**

gas cloud. The reason for this is that in the present case the particles at the leading edge of the expansion have no energy at $t = 0$ and subsequently are accelerated by the particles in the interior of the gas cloud. For $\sigma = 0$, there are sufficient particles at and near the energy corresponding to conditions at $t = r = 0$ to drive the front to the limiting velocity $dR/dt = 2/(\gamma - 1)$. With increase in $\sigma$, there is a decrease in the relative number of particles that have energy at or near the value at $t = r = 0$. Hence, the front is accelerated to lower velocity as $\sigma$ increases.

After long times, the velocity and density are given by [from Eqs. (7) and (12)]

$$v = r/t \tag{13a}$$

$$\rho t^{\sigma+1} = \left(\frac{(\sigma + 1)^{1/2}(\gamma - 1)}{2}\right)^{\sigma+1}(1 - \eta^2)^{1/(\gamma-1)} \tag{13b}$$

where $\eta = (\sigma + 1)^{1/2}[(\gamma - 1)/2](r/t)$. These expressions are in agreement with Eqs. (4) and (5).

### Analytical Solution for $\sigma = 0$ and Uniform Initial Density

Reference 3 has given the characteristics solution for the plane ($\sigma = 0$) expansion into a vacuum of a gas that is initially uniform. This solution has been discussed in greater detail

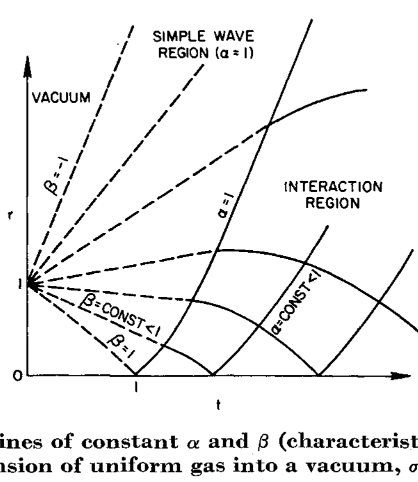


**Fig. 3 Lines of constant $\alpha$ and $\beta$ (characteristics) for expansion of uniform gas into a vacuum, $\sigma = 0$**

in Refs. 5 and 6. The uniform gas originally extends from $r = 0$ to $r = 1$, where $r = 0$ can correspond to a wall or to a plane of symmetry. The expansion is assumed to start at $t = 0$. At first a simple expansion wave is generated which is centered at $r = 1$, $t = 0$. This region is bounded by the lines $\beta = -1$, $\beta = 1$, and $\alpha = 1$ in Fig. 3. The reflection of this wave, at $r = 0$, results in an interaction region, also indicated in Fig. 3. The solution for both regions is discussed herein. In this discussion it is convenient to use the local speed of sound $a = \bar{a}/\bar{a}_0$, instead of $\rho$, as a dependent variable. These are related by, for isentropic flow,

$$\rho = a^{2/(\gamma-1)} \tag{14}$$

In the simple wave region, the dependent variables depend only on $(r - 1)/t$. The solution of Eqs. (3) for $v$ and $a$ then can be shown to be

$$v = \frac{2}{\gamma + 1}\left[1 + \frac{r - 1}{t}\right] \tag{15a}$$

$$a = \frac{2}{\gamma + 1}\left[1 - \frac{\gamma - 1}{2}\frac{r - 1}{t}\right] \tag{15b}$$

The leading edge of the simple wave corresponds to $a = 0$, and its location is given by $R = r = 1 + 2t(\gamma - 1)$. The leading edge moves with velocity $dR/dt = 2/(\gamma - 1)$, as previously noted. The trailing edge of the simple wave is defined by the line $r = (1 - t)$. The fluid between this line and $r = 0$ is uniform at the initial values, $v = 0$, $\rho = a = 1$.

The solution for the interaction region requires the introduction of the Riemann variables:

$$\alpha = a + [(\gamma - 1)/2]v \tag{16a}$$

$$\beta = a - [(\gamma - 1)/2]v \tag{16b}$$

The quantities $\alpha$ and $\beta$ are constant along characteristic lines that propagate with velocity $v + a$ and $v - a$, respectively. These characteristic lines are illustrated in Fig. 3. (In the simple wave region $\alpha = 1$, whereas $\beta$ varies from $-1$ at the leading edge to $+1$ at the trailing edge.) From Eqs. (16) it follows that

$$v = [1/(\gamma - 1)](\alpha - \beta) \tag{17a}$$

$$a = \tfrac{1}{2}(\alpha + \beta) \tag{17b}$$

With these relations, Eqs. (3) can be inverted so that $(r, t)$ are considered functions of $(\alpha,\beta)$. The resulting equations are

$$\frac{\partial r}{\partial \beta} = \frac{1}{\gamma - 1}\left(\frac{\gamma + 1}{2}\alpha + \frac{\gamma - 3}{2}\beta\right)\frac{\partial t}{\partial \beta} \tag{18a}$$

$$\frac{\partial r}{\partial \alpha} = \frac{1}{\gamma - 1}\left(\frac{3 - \gamma}{2}\alpha - \frac{\gamma + 1}{2}\beta\right)\frac{\partial t}{\partial \alpha} \tag{18b}$$

Eliminating $r$ yields a single second-order partial differential equation:

$$\frac{\partial^2 t}{\partial \alpha \partial \beta} + \frac{N}{\alpha + \beta}\left(\frac{\partial t}{\partial \alpha} + \frac{\partial t}{\partial \beta}\right) = 0 \tag{19a}$$

where

$$N = (\gamma + 1)/2(\gamma - 1) \tag{19b}$$

The boundary between the simple wave region and the interaction region (line $\alpha = 1$ in Fig. 3) is defined by[3, 5, 6]

$$t(1,\beta) = [2/(1 + \beta)]^N \tag{20}$$

where $\beta$ varies from 1 to $-1$ as $t$ goes from 1 to $\infty$. An explicit expression for $t(\alpha,\beta)$ in the interaction region is found

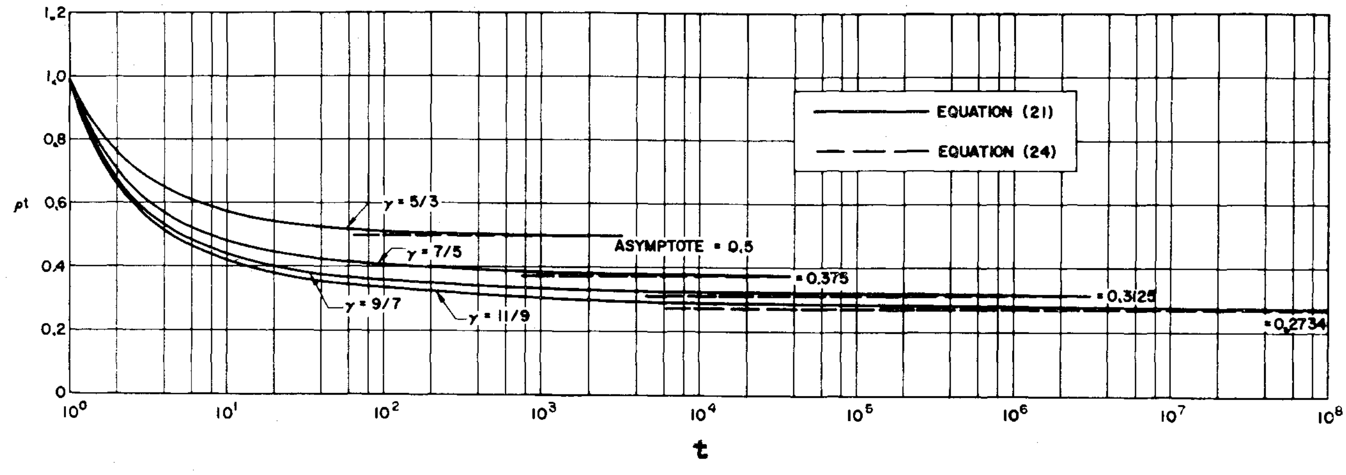


**Fig. 4 Variation of density with time, at $r = 0$, for plane ($\sigma = 0$) expansion of initially uniform gas into vacuum**

by integrating Eq. (19a) with the initial conditions given by Eq. (20) and equals

$$t(\alpha,\beta) = \left(\frac{2}{\alpha+\beta}\right)^N F\left(1 - N,N,1; -\frac{(1-\alpha)(1-\beta)}{2(\alpha+\beta)}\right) \tag{21}$$

Here, $F(\ )$ is the hypergeometric function. If $N$ is a positive integer, the hypergeometric function in Eq. (21) reduces to a polynomial of degree $N - 1$. In particular, for $N$ a positive integer,

$$F(1 - N,N,1;z) = 1 + \sum_{i=1}^{N-1} \frac{\Gamma(N+i)}{\Gamma(N-i)[\Gamma(i+1)]^2}(-z)^i \tag{22}$$

where $\Gamma(\ )$ is the gamma function $[(\Gamma(N+1) = N\Gamma(N), \Gamma(1) = 1, \Gamma(\frac{1}{2}) = (\pi)^{1/2}]$. Equations (18a) and (18b) then can be integrated to give $r(\alpha,\beta)$. Thus, for $N$ an integer, an analytic solution for the interaction region is obtained. Note, from Eq. (19b), that $\gamma = (2N + 1)/(2N - 1)$, so that $\gamma = 3, \frac{5}{3}, \frac{7}{5} \ldots$ for $N = 1,2,3, \ldots$.

The variation with time of the fluid properties at $r = 0$ can be found from Eq. (21). At $r = 0$, $\alpha = \beta = a$, so that Eq. (21) relates $t$ and $a$ (or $\rho,p,T$) for $t \geq 1$. The variation of $\rho$ with $t$, at $r = 0$, is plotted in Fig. 4 for $\gamma = \frac{5}{3}, \frac{7}{5}, \frac{9}{7}, \frac{11}{9}$. The asymptotic solution, for long times, is indicated by a dashed line in this figure. It is seen that the flow approaches the asymptotic solution more slowly as $\gamma$ decreases.

The entire flow field, after long times, also can be described. The velocity field is given by $v = r/t$ [Eq. (4)]. The density distribution and other state properties are found from Eq. (21). Note that $\alpha + \beta = 2a$ approaches zero as $t \to \infty$. Hence $(1 - \alpha)(1 - \beta)/2(\alpha + \beta) \to \infty$ as $t \to \infty$, provided $\alpha < 1$, and only the last term need be retained in the polynomial expansion for $F(\ )$ in Eq. (22). This results in the following expression for $\rho$:

$$\rho t = \frac{(\gamma-1)}{\pi^{1/2}}\frac{\Gamma(N+1/2)}{\Gamma(N)} \times \left[1 - \left(\frac{\gamma-1}{2}\frac{r}{t}\right)^2\right]^{(3-\gamma)/2(\gamma-1)} \tag{23}$$

Equation (23) is valid for $N$ an integer, $t \to \infty$, and $\alpha < 1$.

After long times, the characteristic line $\alpha = 1$ propagates with velocity $2/(\gamma - 1)$ [from Eq. (16a) with $\alpha = 1$, $a = 0$] and thus is parallel to the leading edge of the simple wave ($\beta = -1$ in Fig. 3). With increasing time, the mass contained between these lines becomes negligible (i.e., the simple wave region becomes negligible), and the leading edge of the expansion can be approximated by $R = 2t/(\gamma - 1)$. Again defining $\eta \equiv r/R$, Eq. (23) then becomes

$$\rho t = \frac{(\gamma-1)}{\pi^{1/2}}\frac{\Gamma(N+1/2)}{\Gamma(N)}(1 - \eta^2)^{(3-\gamma)/2(\gamma-1)} \tag{24}$$

which may be considered valid for $t \to \infty$ and $0 \leq \eta \leq 1$. Equation (24) is quite similar in form to Eq. (13b) with $\sigma = 0$. The asymptotic solution noted in Fig. 4 is obtained from Eq. (24) with $\eta = 0$.

### Approximate Asymptotic Solution for Uniform Initial Density

Equation (24) gives the asymptotic nature of the flow, as $t \to \infty$, for a plane expansion ($\sigma = 0$) of an initially uniform gas cloud. It would be of interest to develop corresponding expressions for cylindrical and spherical expansions of an initially uniform gas cloud. Approximate expressions, which satisfy conservation of mass and energy, now will be developed.

Equations (13b) and (24) suggest that a suitable form for $\rho$, as $t \to \infty$, is

$$\rho t^{\sigma+1} = D(1 - \eta^2)^B \tag{25}$$

where $B$ and $D$ are constants and $\eta = r/R$. Since an initially uniform gas cloud is being considered, $R$ will be taken as

$$R = [2/(\gamma - 1)]t \tag{26}$$

The constants $B$ and $D$ can be found from conservation of mass and energy. The latter can be expressed as

$$\frac{1}{\sigma+1} = R^{\sigma+1}\int_0^1 \rho\eta^\sigma d\eta \tag{27a}$$

$$\frac{1}{\gamma(\gamma-1)(\sigma+1)} = R^{\sigma+1}\int_0^1 \rho\frac{v^2}{2}\eta^\sigma d\eta \tag{27b}$$

Equation (27a) equates the mass at $t = 0$ to the mass at $t \to \infty$, and Eq. (27b) equates the internal energy at $t = 0$ to the kinetic energy at $t \to \infty$. (The internal energy $\to 0$ as $t \to \infty$.) Substituting Eqs. (25) and (26) in Eqs. (27), with $v = r/t$ gives

$$B = \frac{\sigma+1}{2}\left(\frac{\gamma+1}{\gamma-1}\right) - 1 = (\sigma+1)N - 1 \tag{28a}$$

$$D = \frac{2}{\sigma+1}\left(\frac{\gamma-1}{2}\right)^{\sigma+1}\frac{\Gamma\{B+1+[(\sigma+1)/2]\}}{\Gamma(B+1)\Gamma[(\sigma+1)/2]} \tag{28b}$$

For $\sigma = 1$ and $N$ and integer, $D = (\gamma^2 - 1)/4$. Numerical

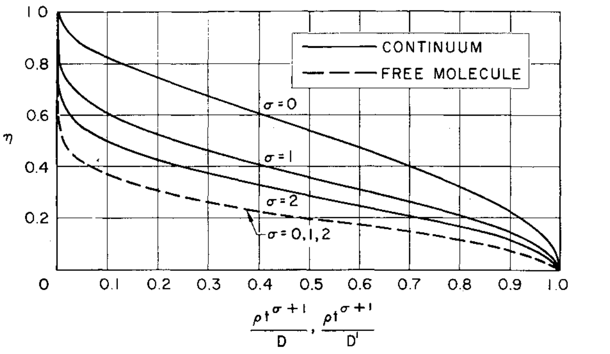


**Fig. 5 Variation of density with $\eta$ for continuum and free molecular expansion of initially uniform gas ($t \to \infty$, $\gamma = \frac{7}{5}$)**

values for $B$ and $D$ are given in Table 1 for $\sigma = 0, 1, 2$ and $\gamma = \frac{5}{3}, \frac{7}{5}, \frac{9}{7}, \frac{11}{9}$, and $\frac{13}{11}$.

Equations (25) and (28) are in exact agreement with Eq. (24) for $\sigma = 0$ and $N$ an integer. Since Eqs. (25) and (28) are an approximate integral method solution, they may be considered valid for $N$ not an integer, as well as for $\sigma = 0$, 1, 2. A similar solution has been presented in Ref. 6, among number of other approximate solutions.

The variation of density with $\eta$ is given in Fig. 5 for $\gamma = \frac{7}{5}$ and $\sigma = 0, 1, 2$. It is seen that, as $\sigma$ increases, the density decreases more rapidly with $\eta$. The variation of density with $\eta$ is given in Fig. 6 for $\sigma = 1$ and $\gamma = \frac{5}{3}, \frac{7}{5}$, and $\frac{9}{7}$. The density decreases more rapidly with $\eta$ as $\gamma$ decreases.

### Free Molecular Expansion

If the mean free path§ of the molecules in a gas is of order $\bar{R}_0$ or larger, the expansion may be treated as a free molecule flow. That is, collisions between molecules can be neglected, and each molecule moves with a constant velocity throughout the expansion. This case has been discussed in Ref. 4 for various initial density distributions. The free molecular expansion of initially uniform gas clouds is discussed herein for $t \to \infty$, and the results are compared with those from continuum theory.

Consider an elemental mass $\bar{m}_0$ of particles to be located at $\bar{r}_0$ at $\bar{t} = 0$ and to have a Maxwellian distribution of velocities. The mass of particles having velocities between $\bar{v}$ and $\bar{v} + \Delta\bar{v}$ is denoted by $\Delta\bar{m}_0$ and equals

$$\Delta\bar{m}_0/\bar{m}_0 = 4\pi(h/\pi)^{3/2}e^{-h\bar{v}^2}\bar{v}^2\Delta\bar{v} \tag{29}$$

where $h = \gamma/(2\bar{a}_0^2)$. Assume that the container that confines $\bar{m}_0$ is removed at $\bar{t} = 0$. At a later time $\bar{t}$, the particles in the velocity range $\bar{v}$ to $\bar{v} + \Delta\bar{v}$ will be contained in the volume bounded by $\bar{r} = \bar{r}_0 + \bar{v}\bar{t}$ and $\bar{r} = \bar{r}_0 + (\bar{v} + \Delta\bar{v})\bar{t}$. This volume has the magnitude $\Delta\bar{V} = 4\pi\bar{v}^2\bar{t}^3\,\Delta\bar{v}$. The density of the particles $\bar{\rho}(\bar{r},\bar{t}) \equiv \Delta\bar{m}_0/\Delta\bar{V}$ is then

$$\bar{\rho}(\bar{r},\bar{t}) = \frac{(h/\pi)^{3/2}e^{-h[(\bar{r}-\bar{r}_0)/\bar{t}]^2}\,\bar{m}_0}{\bar{t}^3} \tag{30}$$

Equation (30) gives the density at $(r,\bar{t})$ due to an initial elemental mass $\bar{m}_0$ at $(\bar{r}_0, 0)$. The expansion of a gas, with an arbitrary initial distribution of density, can be found by the linear superposition of the elementary solution given by Eq. (30) which corresponds to a spherical ($\sigma = 2$) expansion of an elemental three-dimensional mass source. A similar expression can be found for an elemental line source ($\sigma = 1$) or plane source ($\sigma = 0$) located at $(\bar{r}_0, 0)$. The result, valid for all $\sigma$, is

$$\bar{\rho}(\bar{r},\bar{t}) = \frac{(h/\pi)^{(\sigma+1)/2}\,e^{-h[(\bar{r}-\bar{r}_0)/\bar{t}]^2}}{\bar{t}^{\sigma+1}}\,\bar{m}_0 \tag{31}$$

where $\bar{m}_0$ equals the initial elemental mass for $\sigma = 2$, the elemental mass per unit length for $\sigma = 1$, and the elemental mass per unit surface area for $\sigma = 0$ (assuming a symmetrical expansion).

Now, consider the expansion of an initially uniform gas of radius $\bar{R}_0$ and $\sigma = 0, 1, 2$. After long times, the flow may be considered as having originated from an elemental source. The solution for $t \to \infty$ then is given by Eq. (31), which can be put in nondimensional form by introducing $r = \bar{r}/\bar{R}_0$, $v = \bar{v}/\bar{a}_0$, $t = \bar{t}\bar{a}_0/\bar{R}_0$, etc. In the present case, define $\eta = (\gamma - 1) \times r/2t$ so that $\eta = 1$ corresponds to the leading edge for the corresponding continuum expansion. (There is no leading edge for the free molecule expansion, since $r \to \infty$ as $\rho \to 0$.) Equation (31) then becomes

$$\rho t^{\sigma+1} = D'e^{-B'\eta^2} \tag{32}$$

where

$$B' = 2\gamma/(\gamma - 1)^2$$

$$D' = (2\gamma/\pi)^{1/2}, \qquad \gamma/2, \qquad (\gamma/3)(2\gamma/\pi)^{1/2} \qquad \sigma = 0, 1, 2$$

Note that $B'$ is independent of $\sigma$. Numerical values for $B'$ and $D'$ are tabulated in Table 1. Plots of $\rho t^{\sigma+1}$ vs $\eta$ are presented in Figs. 5 and 6. Although the form of Eq. (32) is different from that of Eq. (25), the variation of $\rho t^{\sigma+1}$ with $\eta$ is similar for the two equations, except near the leading edge of the expansion (see Figs. 5 and 6). For $\eta = 0$, Eq. (24) becomes $\rho t^{\sigma+1} = D$, and Eq. (32) becomes $\rho t^{\sigma+1} = D'$, so that $D$ and $D'$ determine the variation of $\rho$ with $t$ at $\eta = 0$. From Table 1 it is seen that, for each $\sigma$ and $\gamma$, $D < D'$ so that the density at the axis decays more rapidly, with time, for continuum flow. The difference in the expansion rates becomes more pronounced as $\gamma$ approaches 1.

The latter result can be understood more readily by considering the distribution of mass within the free molecular expansion. Let $\psi(\eta)$ be the fraction of the total mass which is contained between the ordinate $\eta$ and the axis. Then, using Eq. (32),

$$\psi(\eta) \equiv \int_0^r \rho r^\sigma\,dr \Big/ \int_0^\infty \rho r^\sigma\,dr = \operatorname{erf}[(B')^{1/2}\eta] \qquad (\sigma = 0) \quad \text{(33a)}$$

$$= 1 - e^{-B'\eta^2} \qquad (\sigma = 1) \quad \text{(33b)}$$

$$= \operatorname{erf}[(B')^{1/2}\eta] - \frac{2[(B')^{1/2}\eta]}{\pi^{1/2}}\,e^{-B'\eta^2} \qquad (\sigma = 2) \quad \text{(33c)}$$

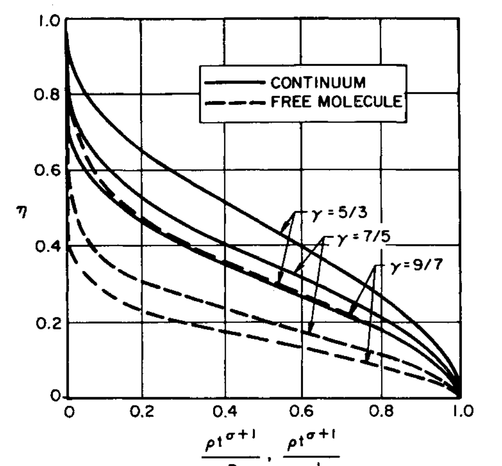


**Fig. 6 Variation of density with $\eta$ for continuum and free molecular expansion of initial uniform gas ($t \to \infty$, $\sigma = 1$)**

§ An approximate expression for the mean free path is $\bar{l} = \bar{\nu}_0/\bar{a}_0$, where $\bar{\nu}_0$ is kinematic viscosity. Free molecular flow theory applies when $\bar{l}/\bar{R}_0 = \bar{\nu}_0/\bar{a}_0\bar{R}_0 > 0(1)$.

**Table 1 Constants defining continuum and free molecular expansion of initially uniform gas clouds as $t \to \infty$ [see Eqs. (28) and (32)]**

| | $\sigma = 0$ | | | | $\sigma = 1$ | | | | $\sigma = 2$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\gamma$ | $B$ | $D$ | $B'$ | $D'$ | $B$ | $D$ | $B'$ | $D'$ | $B$ | $D$ | $B'$ | $D'$ |
| 5/3 | 1 | 0.5 | 7.5 | 1.03 | 3 | 0.444 | 7.5 | 0.833 | 5 | 0.434 | 7.5 | 0.572 |
| 7/5 | 2 | 0.375 | 17.5 | 0.944 | 5 | 0.24 | 17.5 | 0.7 | 8 | 0.169 | 17.5 | 0.441 |
| 9/7 | 3 | 0.313 | 31.5 | 0.905 | 7 | 0.163 | 31.5 | 0.643 | 11 | 0.094 | 31.5 | 0.388 |
| 11/9 | 4 | 0.273 | 49.5 | 0.882 | 9 | 0.123 | 49.5 | 0.611 | 14 | 0.061 | 49.5 | 0.359 |
| 13/11 | 5 | 0.246 | 71.5 | 0.867 | 11 | 0.099 | 71.5 | 0.591 | 17 | 0.044 | 71.5 | 0.342 |

where erf( ) is the error function. The effective leading edge of the free molecular expansion might be taken to correspond to $\psi(\eta) = 0.9$, 0.99, or 0.999, etc. The values of $(B')^{1/2}\eta$ which correspond to $\psi(\eta) = 0.9$, 0.99, and 0.999 are given in Table 2. Let $(B')^{1/2}\eta \approx 2$ be taken as a characteristic value defining the effective leading edge of the free molecular expansion, and let the corresponding value of $r$ be denoted by $R_{\text{eff}}$. Then $R_{\text{eff}}/t \approx 2(2/\gamma)^{1/2}$. This may be compared with $R/t = 2/(\gamma - 1)$ for the continuum case, provided $\gamma \neq 1$.‖ Hence, the continuum expansion spreads more rapidly, particularly for $\gamma$ near 1.

## Hypersonic Jet

As noted in Ref. 5, the lateral expansion of a steady hypersonic jet, bounded by a vacuum, can be reduced to an equivalent unsteady expansion. This equivalence now is discussed.

Consider a nozzle to exhaust a hypersonic stream into a vacuum. Let $\bar{u}$ be the velocity in the streamwise $\bar{x}$ direction, $\bar{u}_0$ be the value of $\bar{u}$ at the nozzle exit, $M_0 = \bar{u}_0/\bar{a}_0$, and $\delta$ be a characteristic slope of the streamlines. If the dependent variables have the form

$$\bar{v}/\bar{u}_0 = 0(\delta) \qquad \bar{\rho}/\bar{\rho}_0 = 0(1) \qquad \bar{u}/\bar{u}_0 = 1 + 0(\delta^2) \quad (34)$$

then the continuity and $\bar{v}$ momentum equations reduce to Eqs. (1) (neglecting terms of order $\delta^2$ compared with 1) with $\bar{t}$ replaced by $\bar{x}/\bar{u}_0$. These equations can be solved for $\bar{v}$ and $\bar{p}$ independent of the $\bar{x}$ momentum equation. Bernoulli's equation can then be used to find $\bar{u}$ (which equals $\bar{u}_0$ to order $\delta^2$). The characteristic slope may be taken equal to that of the leading edge of the expansion after long times

$$\delta = [2/(\gamma - 1)](1/M_0) \quad (35)$$

The foregoing solution is self-consistent when $M_0^2 \gg 1$ (provided $\gamma$ is not nearly equal to 1) and hence applies to hypersonic jets. Thus, the previous unsteady solutions for $\sigma = 0, 1$ apply to two-dimensional ($\sigma = 0$) and axisymmetric ($\sigma = 1$) hypersonic jets. The variable $\bar{t}$ in the former is replaced by $\bar{x}/\bar{u}_0$ in the latter. If a nondimensional $x = \bar{x}/\bar{R}_0$ is introduced, then $t$ is replaced by $x/M_0$ in the nondimensional equations.

## Numerical Solutions

The equations of motion for an initially uniform hypersonic axisymmetric jet exhansting into a vacuum have been integrated numerically for $M_0 = 10, 15$, $\gamma = 1.4$, and $M_0 = 10$, $\gamma = 1.22$. The computations were performed by A. B. Troesch and R. F. Kramer using an existing digital computer program. The density along the axis of the jet was found as a function of distance and is plotted in Fig. 7. The approximate asymptotic solution, previously presented [Eqs. (28)], also is noted in Fig. 7. It was not possible to continue the machine solution until the asymptotic form $\rho(x/M_0)^{\sigma+1} = D$ was attained.

Attempts to extrapolate the machine results analytically to very large distances, for direct comparison with the approximate theory, also met with some difficulty, the answer depending on the extrapolation scheme used. For the $M_0 = 10, 15$, $\gamma = 1.4$ case, however, some confidence can be placed in the extrapolated value $D = 0.22$, which can be compared to the approximate value $D = 0.24$ given in Table 1 for $\gamma = 1.4$, $\sigma = 1$. As expected, the results for $M_0 = 10, 15$

‖ A more precise comparison can be made by computing $\psi(\eta)$ for the continuum flow. This can be done in closed form for $\sigma = 1$. In particular, for $\sigma = 1$ and continuum flow, Eqs. (25) and (28a) show $\psi(\eta) = 1 - (1 - \eta^2)^{(\gamma + 1)/(\gamma - 1)}$. For a given value of $\epsilon \equiv 1 - \psi(\eta)$ and $\sigma = 1$, the corresponding values of $\eta$ for the free molecular and continuum flows are related by $\eta_{FM}/\eta_{\text{cont}} = (\gamma - 1)(-\ln\epsilon)^{1/2}/(2\gamma)^{1/2}[1 - \epsilon^{(\gamma - 1)(\gamma + 1)}]^{1/2}$. For $\epsilon \ll 1$ and $\gamma \neq 1$, the latter gives $(R_{\text{eff}})_{FM}/(R_{\text{eff}})_{\text{cont}} = (\gamma - 1)(-\ln\epsilon)^{1/2}/(2\gamma)^{1/2}$, which is essentially the comparison given in the body of the report.

**Table 2 Effective leading edge of free molecular expansion**

| | $(B')^{1/2}\eta$ | | |
|---|---|---|---|
| $\psi(\eta)$ | $\sigma = 0$ | $\sigma = 1$ | $\sigma = 2$ |
| 0.9 | 1.16 | 1.52 | 1.77 |
| 0.99 | 1.82 | 2.15 | 2.40 |
| 0.999 | 2.33 | 2.63 | 2.85 |

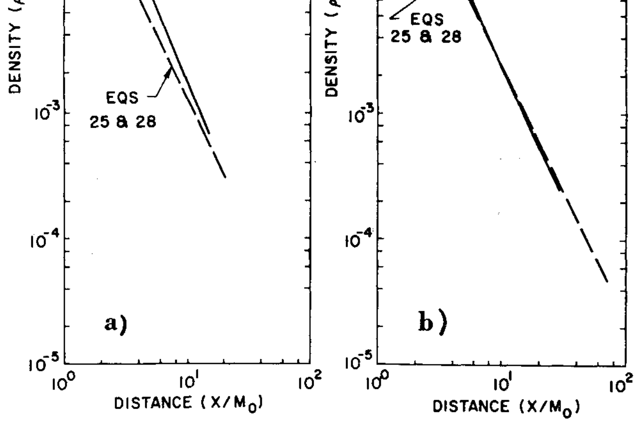


**Fig. 7 a) Variation of density with distance along the axis ($r = 0$) of axisymmetric steady hypersonic jet ($M_0 = 10$; $\gamma = 1.22$) expanding into vacuum; continuum flow. b) Variation of density with distance along the axis ($r = 0$) of axisymmetric steady hypersonic jet ($M_0 = 10, 15$; $\gamma = 1.4$) expanding into vacuum; continuum flow**

and $\gamma = 1.4$ were essentially independent of Mach number and are plotted as a single curve in Fig. 7b.

Reference 4 gives numerical results for the density at the center of an initially uniform spherical gas cloud expanding to a vacuum for $\gamma = \frac{5}{3}$. The results are presented by a plot of $\rho$ vs $t$ which yields $D = 0.38$. The latter value can be compared with the approximate value $D = 0.434$ given in Table 1 for $\sigma = 2$, $\gamma = \frac{5}{3}$. However, it should be noted that the initial portion of the continuum curve of $\rho$ vs $t$ in Ref. 4 obviously is incorrect, since the numerical solution for $\rho$ vs $t$ approaches the asymptotic solution from below rather than from above (compare with Fig. 7). This casts some doubt on the correctness of the resulting value $D = 0.38$.

In general, it appears that the value of $D$ given by the approximate solution [Eq. (28)] tends to be somewhat too large for $\sigma = 1, 2$, although it is exact for $\sigma = 0$ and $N$ an integer. The two numerical solutions noted herein indicated that $D$ is too large by about 10%. Further numerical solutions are required to define better the accuracy of the approximate solution.#

## References

[1] Sedov, L. I., *Similarity and Dimensional Methods in Mechanics* (Academic Press, New York, 1959), pp. 271–281.

[2] Keller, J. B., "Spherical, cylindrical, and one dimensional gas flows," Quart. Appl. Math. **14**, 171–184 (1956).

[3] Courant, R. and Friedricks, K. O., *Supersonic Flow and Shock Waves* (Interscience Publishers, New York, 1948), pp. 191–195.

[4] Molmud, P., "Expansion of a rarified gas cloud into a vacuum," Phys. Fluids **3**, 362–366 (1960).

[5] Greifinger, C. and Cole, J., "One dimensional expansion of a finite mass of gas into vacuum," Rand Rept. P2008 (June 6, 1960).

[6] Stanyukovich, K. P., *Unsteady Motion of Continuous Media* (Pergamon Press, New York, 1960), pp. 147–163, 498–506.

[7] Greenspan, H. P. and Butler, D. S., "On the expansion of a gas into vacuum," J. Fluid Mech. **13**, 101–119 (1962).

# After completion of the present report, the authors communicated with C. Greifinger, of Rand Corporation, who has obtained numerical solutions for the continuum expansion of an initially uniform gas into a vacuum. These results are as yet unpublished. However, Dr. Greifinger has compared the values of $D$, in Table 1, with his results. He stated that, for $\sigma = 1$, the values of $D$ for $\gamma = \frac{11}{9}$ and $\gamma = 1.4$ are about 10% and 20% high, respectively. For $\sigma = 2$, he has stated that the values of $D$ for $\gamma = \frac{11}{9}$ and $\gamma = \frac{5}{3}$ are about 50% and 100% high, respectively. Thus the approximate solution of the present report appears to become more in error as $\sigma$ and $\gamma$ increase. However, for hypersonic rocket exhausts ($\sigma = 1$, $\gamma \approx 1.2$), the approximate solution should be correct to within 10%. The publication of Dr. Greifinger's results will permit a more accurate evaluation of the approximate solution and might provide a basis for improvement.

MARCH 1963 AIAA JOURNAL VOL. 1, NO. 3

# Powered Flight Trajectories of Rockets under Oriented Constant Thrust

Chong-Hung Zee*
*Curtiss-Wright Corporation, Wood-Ridge, N. J.*

**The second-order nonlinear differential equations of motion in the case of a rocket in drag-free powered flight under a thrust of constant magnitude and fixed orientation are solved by series expansions developed to the seventh power of the independent variable "time." The coefficients of each power of "time" are in terms of the preceding ones and, consequently, in terms of the initial conditions. The truncation errors of the series are estimated; hence their accuracy can be evaluated. The case of oriented constant thrust acceleration also is included in the present analysis.**

## Nomenclature

$A$ = $\dot{m}\bar{c}/M_0 g_0$
$a_0, a_1, a_2, \ldots$ = constants to be determined
$B$ = $(\dot{m}/M_0)(r_0/g_0)^{1/2}$
$b_0, b_1, b_2, \ldots$ = constant to be determined
$C$ = $(d\rho/d\tau)_0$
$C'$ = $(dr/dt)_0$
$\bar{c}$ = effective average exhaust velocity of the jet
$D$ = $(d\theta/d\tau)_0$
$D'$ = $(d\theta/dt)_0$
$g_0$ = gravitational constant at distance $r_0$ from the center of attraction
$M$ = $\cos\psi$
$M_0$ = mass of rocket at the beginning of thrusting
$\dot{m}$ = constant flow rate of propellant mass
$N$ = $\sin\psi$
$r$ = distance between the rocket and the center of attraction at any time $t$
$r_0$ = distance between the rocket and the center of attraction at $t = 0$
$t$ = time
$V_x$ = velocity component in the $X$ direction
$V_y$ = velocity component in the $Y$ direction
$X$ = coordinate (origin at $r = r_0$, $\theta = 0$)
$Y$ = coordinate (origin at $r = r_0$, $\theta = 0$)
$Rn1, Rn2, \ldots$ = remainder of truncated series
$\theta$ = angle between radius vectors $r_0$ and $r$
$\xi_1, \xi_2, \ldots, \xi_{1m}, \xi_{2m}$ = values of $\tau$ between the interval 0 and $\tau$
$\rho$ = $r/r_0$
$\tau$ = $(g_0/r_0)^{1/2}t$
$\psi$ = angle between the thrust vector and the radius vector $r_0$

Received by ARS August 15, 1962; revision received January 14, 1963.

* Project Engineer, Engineering Department, Wright Aeronautical Division.